Topological methods

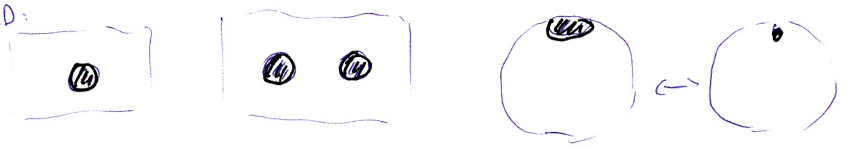
M. Nakahara: Geometry, Topology and Physics, 2003

- investigation of topological structure of manifold
- number of "holes" of different dimensionality
- ignoring metric properties of manifolds

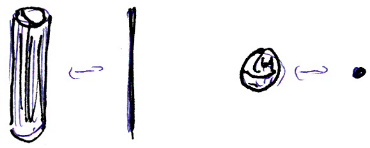
"holes:"

cut out

2D:

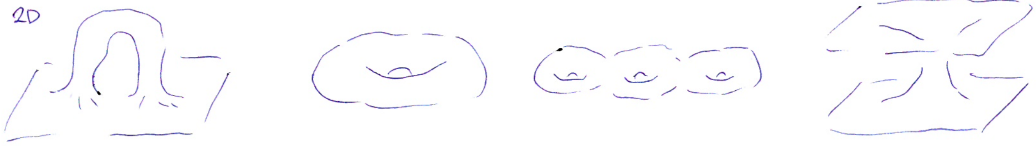


3D



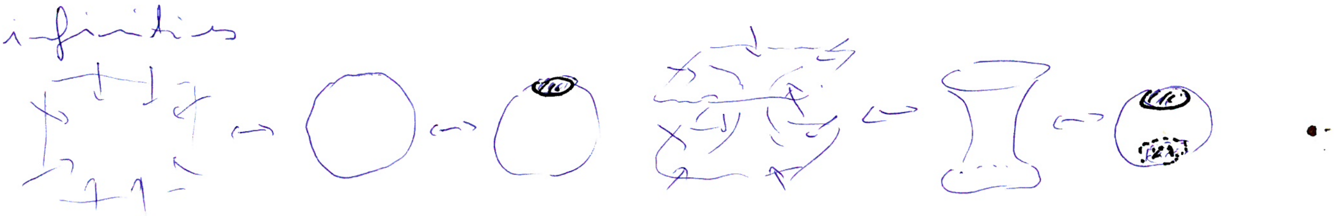
intrinsic holes

2D



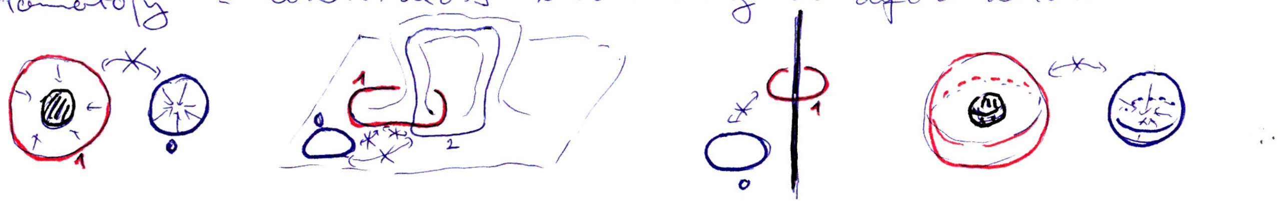
3D

infinities

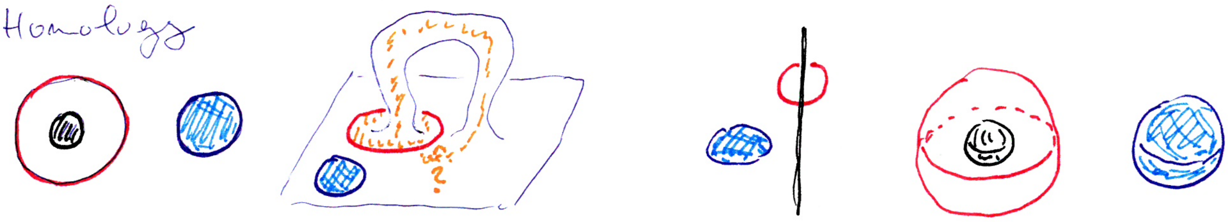


how to recognize holes?

Homotopy = "continuous shrinking" or deformation



Homology



testing using simple submanifolds which
can/cannot be filled

spheres filled by balls (+ combinations of spheres)

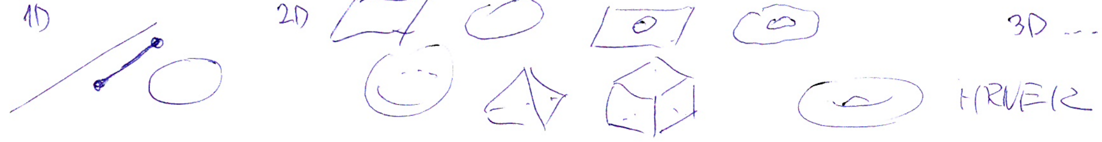
simplexes (gener. of segment and triangle)
and simplical complexes (combinations of simplexes)

compact manif. without boundary (cycle) $\Sigma \quad \partial\Sigma = \emptyset$
can be filled if it is a boundary $\Sigma = \partial\Omega$

Topological characterization of a manifold

- properties of manifolds up to homeomorphism
  (continuous with continuous inverse)

examples



topological invariants

- characteristics which can distinguish non-homeomorph.
- don't have complete set of invariants

Simplicial homology - brief overview
~ triangulation

(geometrical) simplex in $\mathbb{R}^m$

σ^0 σ^1 σ^2 σ^3 σ^4 ...

r-simplex: (unordered)

$\sigma^r = \langle p_0 p_1 \dots p_r \rangle \quad |\sigma^r| = \{ x \in \mathbb{R}^m \quad \sum_{i=0}^{r} t_i p_i \quad t_i \geq 0 \quad \sum_{i=0}^{r} t_i = 1 \}$

p_i vertex of σ^r ← geometr. independent (does not lay in subplane)

q-face of σ^r

- q-simplex ϱ^q given by q vertices of σ^r

$\varrho^q \leq \sigma^r \qquad q \leq r$

simplicial complex K

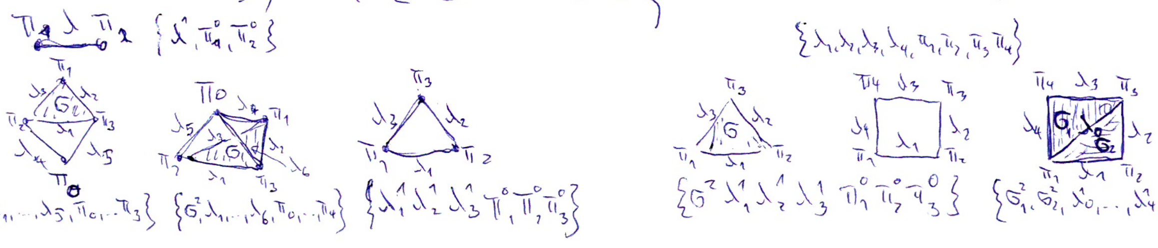
K - finite set of simplices ~~"align"~~ "fitted together"

1) $\sigma \in K \quad \varrho \leq \sigma \Rightarrow \varrho \in K$

2) $\sigma, \varrho \in K$ $\tau = \sigma \cap \varrho$ $\Rightarrow$

a) $\tau = \emptyset$ or

b) $\tau \leq \sigma \quad \tau \leq \varrho$ $\qquad \tau$ is simplex



$\{\lambda^1, \pi_1^0, \pi_2^0\}$ $\qquad \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \pi_1, \pi_2, \pi_3, \pi_4\}$

$\{\sigma^2, \lambda_1, \dots, \lambda_5, \pi_0, \dots, \pi_3\}$ $\{\sigma^2, \lambda_1, \dots, \lambda_6, \pi_0, \dots, \pi_4\}$ $\{\lambda_1^1 \lambda_2^1 \lambda_3^1 \pi_1^0 \pi_2^0 \pi_3^0\}$ $\{\sigma^2 \lambda_1^1 \lambda_2^1 \lambda_3^1 \pi_1^0 \pi_2^0 \pi_3^0\}$ $\{\sigma_1^2, \sigma_2^2, \lambda_0, \dots, \lambda_4, \pi_1^0 \dots \pi_4^0\}$

Polyhedron $|K|$

$|K| = \bigcup_{l} |\sigma_l|$

X topological space (manifold)

X is triangulable iff

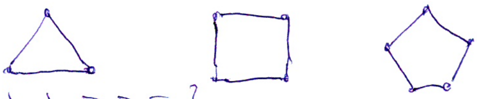
$\exists$ simplicial complex K such that $|K|$ is homeomorphic with X

K is triangulation of X

Ex:

S_1

triang: (segmentation)

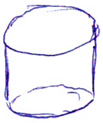


$\{\lambda_1 \lambda_2 \lambda_3 \pi_1 \pi_2 \pi_3\}$

S_2

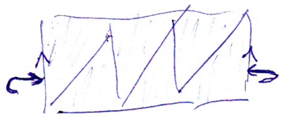
triang.

$\{\sigma_1^2 \sigma_2^2 \sigma_3^2 \sigma_4^2 \lambda_1^1 \dots \lambda_6^1 \pi_1^0 \quad \pi_4^0\}$

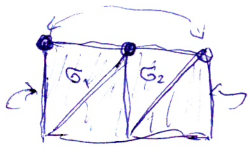


cylinder

triang

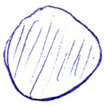


not triang:



intersection $\sigma_1 \cap \sigma_2$ is not simplex

$D = B_1$



triang

$\{\sigma^2, \lambda_1^1, \lambda_2^1, \lambda_3^1, \pi_1, \pi_2, \pi_3\}$ $\{\sigma_1^2, \sigma_2^2, \lambda_0^1 \quad \lambda_4^1, \pi_1 \dots \pi_4\}$

B_3

triang:

$\{\rho^3, \sigma_1^2, \dots \sigma_4^2, \lambda_1^1 \dots \lambda_6^1, \pi_1^0 \dots \pi_4^0\}$ $\{\rho_0^3 \dots \rho_4^3, \sigma_1^2 \quad \sigma_{16}^2, \lambda_1^1 \dots, \lambda_{18}^1, \pi_1^0 \quad \pi_6^0\}$

oriented simplex

$\sigma^r = (p_0 \quad p_r)$ depends only on sign of perm. of vertices

for each unoriented simplex $\sigma = \langle \quad \rangle$ we have two oriented variants

$\sigma = (p's)$ and $-\sigma =$ (odd permut. of p's)

example:

for $\langle p_0 p_1 p_2 \rangle$ we have $(p_0 p_1 p_2) = (p_1 p_2 p_0) = (p_2 p_0 p_1) = -(p_0 p_2 p_1) = -(p_2 p_1 p_0) = -(p_1 p_0 p_3)$

oriented simplicial complex ≡ simpl. complex with selected orientation of each component

r-chain group $C_n(K)$

= Abelian group generated by oriented siplices from K

generated freely up to restriction

$(p_0 \dots p_n) = \text{sign } s \, (p_{s_0} \quad p_{s_n})$ s permutation (essentially free generation over oriented simpl.)

$$C^r = \sum_{j=1}^{I_c} a^j \sigma_j^r \qquad a^j \in \mathbb{Z}$$

r-chain-group over R $C_n(K, R)$

$$C^r = \sum_{j=1}^{I_c} a^j \sigma_j^r \qquad a^j \in R$$

R-module generated by σ_j^r freely up to restr.

$R = \mathbb{Z}$ is "universal" choice

– the most detailed

Abelian groups G

- commuting group.

$x+y=y+x \quad 0 \quad -x \quad nx \quad n\in\mathbb{Z}$

subgr. $H\subset G$ generated by $x_1 \dots x_r$

$n_1x_1 + \quad + n_rx_r \qquad n_j\in\mathbb{Z}$

lin

$x_1 \dots x_r$ lin independ. iff

$n_1x_1 + \quad + n_rx_r = 0 \iff n_j = 0$

G is finitely gener.

- G is gener. by finite number of gen.

G is free Abel. gr. of rank r

G is finit gener by r lin indep. gen.

Cyclic group.

G generated by x

finite cyclic $\quad nx=0 \quad n\in\mathbb{N} \quad \to \mathbb{Z}_n$

Fund. theorem

finitely gener. abelian gr.

$$G \cong \underbrace{\mathbb{Z}\oplus \quad \oplus\mathbb{Z}}_{f}\oplus\mathbb{Z}_{n_1}\oplus \cdot \oplus\mathbb{Z}_{n_c}$$

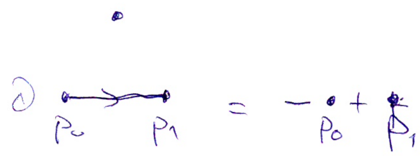
↑ freely gener. subgroup

rank $\quad r = f + c$

Boundary

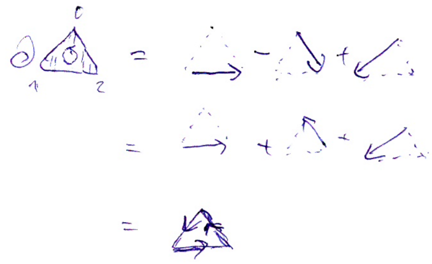
$\partial \pi^0 = 0$

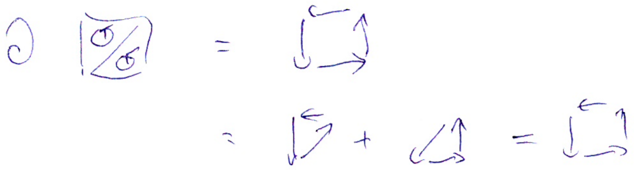
$\partial (p_0 p_1) = p_1 - p_0$



$$\partial \underset{(p_0 \dots p_r)}{\sigma^r} = \sum_{i=0}^{r} (-1)^i (p_0 \dots \underset{\text{skipped}}{\hat{p}_i} \dots p_r)$$

$$\partial (p_0 p_1 p_2) = (p_1 p_2) - (p_0 p_2) + (p_0 p_1)$$
$$= (p_1 p_2) + (p_2 p_0) + (p_0 p_1)$$



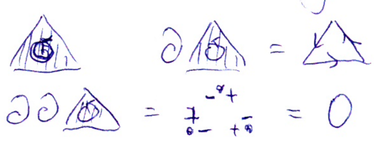


$$\partial : C_r(K) \to C_{r-1}(K)$$

boundary does not have boundary

$\partial\partial = 0$

Ex:



r-cycles

$\sigma \in Z_r(K)$ iff $\partial \sigma = 0$

r-boundaries

$\sigma \in B_r(K)$ iff $\sigma = \partial \varsigma$

$\Downarrow$

$B_r(K) \subset Z_r(K)$

Homology group

$H_r(K) = Z_r(K) / B_r(K)$

cycles eq to boundary

rmk.

$H_r(K;R) = Z_r(K;R) / B_r(K;R)$

c, c' homologous iff $c = c' + \partial e$

$[c] = [c']$

Th $H_r(K)$ are topol. invariants.

Th: K disjoint union of connected complexes

$K = K_1 \cup \quad \cup K_N$

$\Downarrow$ $H_r(K) = H_r(K_1) \oplus \quad \oplus H_r(K_N)$

Th $Z_r(K)$ $B_r(K)$ free generat. Abel. gr.

(subgroups of $C_r(K)$)

$H_r(K)$ does not have to be free

$H_r(K) = \underbrace{\mathbb{Z} \oplus \quad \oplus \mathbb{Z}}_{b_r} \oplus \underbrace{\mathbb{Z}_{M_1} \oplus \quad \oplus \mathbb{Z}_{M_c}}_{c}$ torsion subgroup

$H_r(K, \mathbb{R})$ sensit. only to free part

$H_r(K; \mathbb{R}) = \mathbb{Z} \oplus \quad + \mathbb{Z}$

K_1, K_2 such that $|K_1|, |K_2|$ homeomorphic

$\Rightarrow H_r(K_1) = H_r(K_2)$

Def X topological space which is triangulable by K

we define $H_r(X, R) = H_r(K, R)$

independent of triangul.

Betti numbers & Euler charact.

Betti number

$$b_r(K) = \dim H_r(K;\mathbb{R})$$

its topol. invariant of $|K|$

- does not depend on triang. of X

$b_r(X) \overset{def}{=} b_r(K)$ $\quad K$ triang. of X

Euler character.

$$\chi(X) = \chi(K)$$

$$\chi(K) = \sum_{r=0}^{n} (-1)^r I_r$$

\# vertic. − # edges + # faces − ....

Th:

$$\chi(K) = \sum_{r=0}^{n} (-1)^r b_r(K)$$

proof $\quad \partial : C_r(K) \to C_{r-1}(K)$

$$I_r = \dim C_r(K,\mathbb{R}) \overset{\text{kerting theorem}}{=} \dim \ker \partial_r + \dim \operatorname{img} \partial_r$$

$\dim Z_r(K,\mathbb{R}) = \dim \ker \partial_r$

$\dim B_r(K,\mathbb{R}) = \dim \operatorname{img} \partial_{r+1} \Rightarrow$ $\quad = \dim Z_r(K;\mathbb{R}) + \dim B_{r-1}(K,\mathbb{R})$

$$b_r(K) = \dim H_r(K;\mathbb{R}) = \dim Z_r(K,\mathbb{R})/B_r(K,\mathbb{R})$$

$$= \dim Z_r(K,\mathbb{R}) - \dim B_r(K,\mathbb{R})$$

$$\chi(K) = \sum_{r=0}^{n} (-1)^r I_r = \sum_{r=0}^{n} (-1)^r (\dim Z_r(K,\mathbb{R}) + \dim B_{r-1}(K,\mathbb{R}))$$

$$= \sum_{r=0}^{n} \left[(-1)^r \dim Z_r(K,\mathbb{R}) - (-1)^r \dim B_r(K,\mathbb{R})\right] =$$

$$= \sum_{r=0}^{n} (-1)^r b_r(K)$$

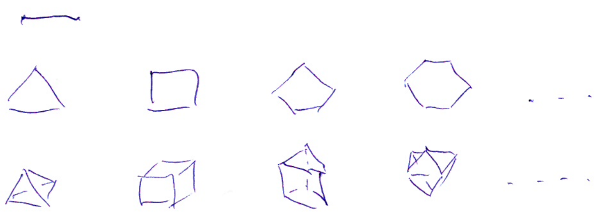
Generalization of Euler formula for "polyhedralization"

- similar construction of complexes but build from convex polyhedra which are homological to simplex

1D

2D

3D



- it can be always refined to simplical complex = it can be triangularized "face by face" and embed into bigger simplical complex which has the same body

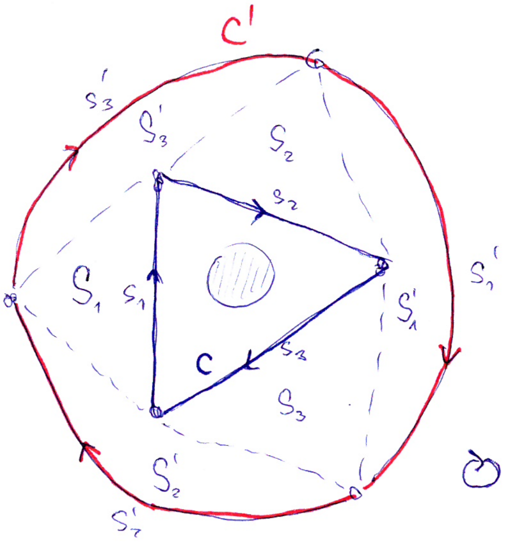
Euler charact. can be defined for polyhedratization

$$\chi(P) = \sum_{r=0}^{n} (-1)^r I_r$$

I_r - number of r-faces

Th: Euler char. of polyhedr. complex is the same as of its triangularization examples

- belongs more to singular homology
- can be understood as examples for simplical homology assuming sufficiently extended simplical complex
  = we allow adding new simplexes



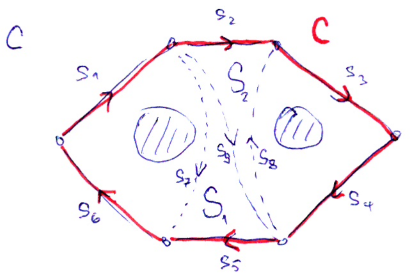
$c = s_1 + s_2 + s_3$ cyklus

$c' = s_1' + s_2' + s_3'$ cyklus

$c' = c + \partial S_1' + \partial S_2' + \partial S_3' + \partial S_1 + \partial S_2 + \partial S_3$

$c' = c + \partial C$ ∂C hranice

$C = S_1 + S_2 + S_3 + S_1' + S_2' + S_3'$



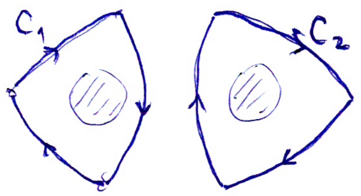
$c = s_1 + s_2 + s_3 + s_4 + s_5 + s_6$

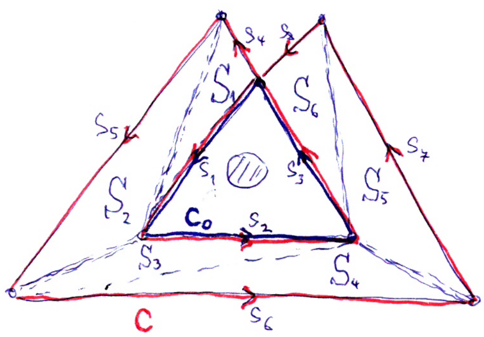
$= (s_1 + s_7 + s_8) + (s_3 + s_4 + s_8)$

$+ (s_9 + s_5 - s_7) + (s_2 - s_8 - s_9)$

$= c_1 + c_2 + \partial S_1 + \partial S_2 =$

$= c_1 + c_2 + \partial C$ $C = S_1 + S_2$





$c_0 = s_1 + s_2 + s_3$

$c = s_1 + s_2 + s_3 + s_4 + s_5 + s_6 + s_7 + s_8$

$s_4 + s_5 + s_6 + s_7 + s_8 = s_1 + s_2 + s_3 + \partial C$

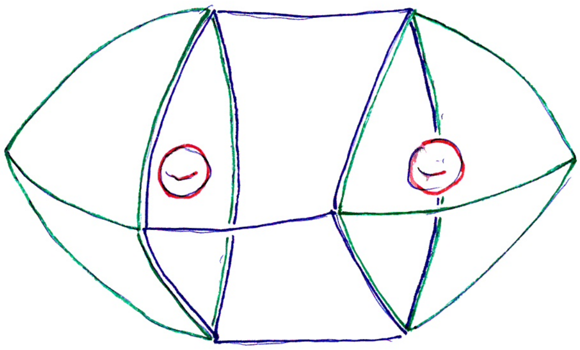
$C = S_1 + S_2 + S_3 + S_4 + S_5 + S_6$

$c = (s_1 + s_2 + s_3) + (s_1 + s_2 + s_3) + \partial C$

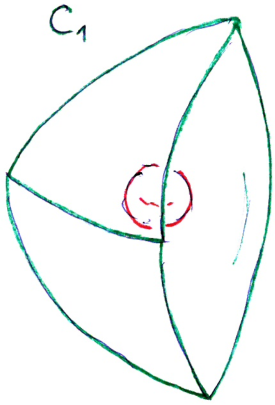
$= 2c_0 + \partial C$

$[c] = 2[c_0]$

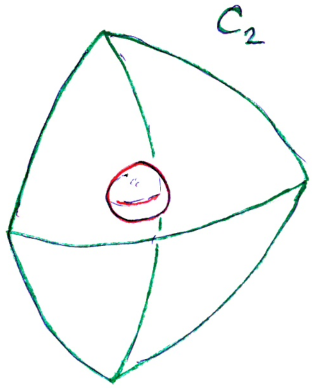
c

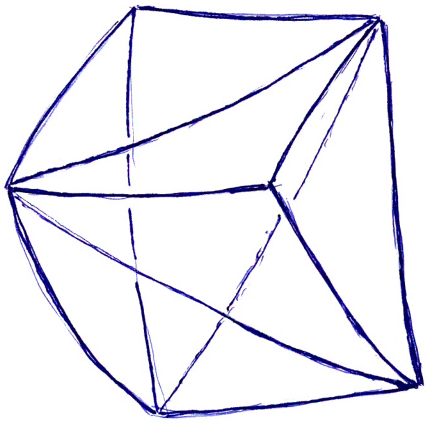


c_1



c_2

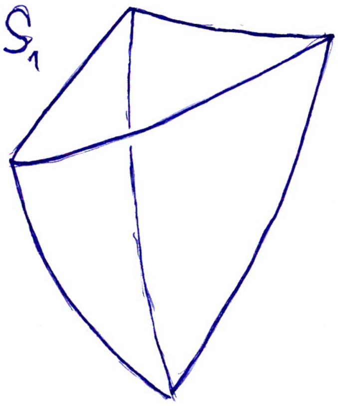




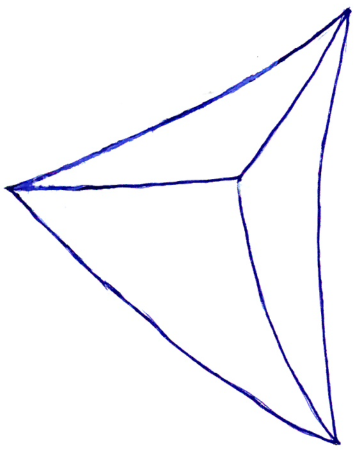
∂C

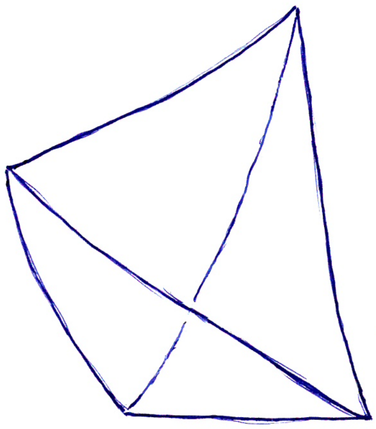
$$C = S_1 + S_2 + S_3$$

S_1



S_2





S_3

$c,\ c_1,\ c_2$ 2-cykly

$C,\ S_1,\ S_2,\ S_3$ 3-chainy

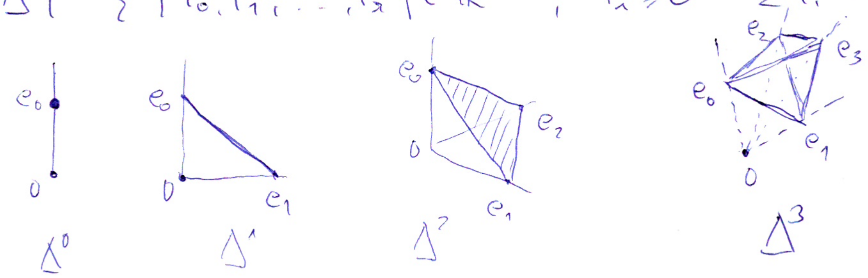
$$c = c_1 + c_2 + \partial C$$

Singular homology

Standard (ordered) simplex

$\Delta^n = [e_0, e_1, \dots, e_n]$

$|\Delta^n| = \{ [t_0, t_1, \dots, t_n] \in \mathbb{R}^{n+1},\ t_i \geq 0 \quad \sum t_i = 1 \}$



Δ^n has information about order of vertices

singular simplex in manifold M

$s^n : \Delta^n \to M$ sufficiently smooth

homomorphism, may be without inversion

n-chains $C_n(M)$

free generated Abelian group generated by finite combination of singular simplices

$c = \sum_{i \text{ finite}} a^i s_i \qquad a^i \in \mathbb{R} \qquad s_i$ n-simplex

not a triangulation - no fitting of simplices

boundary

ith face of a standard simplex Δ^n

$i\text{-face}: \Delta_i^{n-1} \to \Delta^n \qquad [t_0, \dots, t_n] \to [t_0 \dots t_{i-1}, 0, t_{i+1} \dots t_n]$
missing t_i

i-th face of a simplex

$s_i^{n-1} : \Delta_i^{n-1} \to \Delta^n \xrightarrow{s^n} M$

boundary operator

$\partial : C_n(M) \to C_{n-1}(M)$ linear

$\partial s^n = \sum_{i\text{-faces of } s^n} (-1)^i s_i^{n-1}$

$\Rightarrow \partial \partial c = 0$

cycles and boundaries

$C_r(M)$ r-chains

$Z_r(M)$ r-cycles $\partial c = 0$

$Z_r = \{ c \in C_r(M) \, , \ \partial c = 0 \}$

$B_r(M)$ r-boundaries $b = \partial e$

$B_r = \{ b \in C_r(M) \, , \ \exists e \ \ b = \partial e \}$

$\partial\partial \equiv 0 \implies B_r(M) \subset Z_r(M)$

$C_r(M), Z_r(M), B_r(M)$ infinitely dimens.

singular homology

$H_r(M) = Z_r(M) / B_r(M)$

Theorem

M triangulable, K triangularization

$\Downarrow$

$H_r(M) = H_r(K, \mathbb{R})$

$H_r(M)$ finite dimensional for M compact

$b_r = \dim H_r(M)$ - Betti numbers

de Rham cohomology

$C^r(M) = A^r M$ — r-form

$Z^r(M) = A^r_{closed} M$ — closed r-forms (cocycles) $d\omega = 0$

$B^r(M) = A^r_{exact} M$ — exact r-forms (coboundaries) $\omega = d\sigma$

$dd \equiv 0 \Rightarrow B^r(M) \subset Z^r(M)$

$C^r(M), Z^r(M), B^r(M)$ - r-form fields on M
infinitely dimens.

cohomology group

$H^r(M) = Z^r(M) / B^r(M)$

duality of $H_r(M)$ and $H^r(M)$

duality of cycles and cocycles

$$(c, \omega) = \int_c \omega = \sum_i a_i \int_{S_i} \omega \qquad \begin{array}{l} c \in Z_r(M) \\ \omega \in Z^r(M) \end{array}$$

Stokes theorem

$$(c, d\omega) = (\partial c, \omega)$$

induces a duality of homol. and cohomol. gr.

$$\langle [c], [\omega] \rangle = (c, \omega) = \int_c \omega$$

$(c + \partial\tilde{c}, \omega) = (c, \omega) + (\tilde{c}, \underset{0}{d\omega}) = (c, \omega) \Rightarrow$ depends only on $[c]$

$(c, \omega + d\tilde{\omega}) = (c, \omega) + (\underset{0}{\partial c}, \tilde{\omega}) = (c, \omega) \Rightarrow$ depends only on $[\omega]$

de Rham duality

M compact orientable

$\langle \, , \rangle$ bilinear nondegenerate

$\Rightarrow H^r(M) \cong H_r(M)$ of a finite dim b_r

Theorem

$\exists$ frame of cycles $\{c_i\}$ such that

$\{[c_i]\}$ is a frame in $H_k(M)$, i.e. $[c_i] \underset{i \neq j}{\neq} [c_j]$

let $\omega \in C(M)$ (is closed)

ω is exact ($\exists \sigma$ $\omega = d\sigma$) $\iff$

$\forall i$ $\langle [c_i], [\omega] \rangle = 0$

i.e. $\int_{c_i} \omega = 0$

indeed: ω exact means $[\omega] = 0$

$\langle [c_i], [\omega] \rangle$ are components of $[\omega]$

let a_i $a_j \in \mathbb{R}$

$\exists$ ω $d\omega = 0$ $\int_{c_i} \omega = a_i$

proof. let $[\sigma^j]$ is dual frame

$\langle [c_i], [\sigma^j] \rangle = \delta_i^j$

then $[\omega] = \sum_j a_j [\sigma^j]$ satisfies $\langle [c_i], [\omega] \rangle = a_i$

ω is a representant of $[\omega]$